Classical and Quantum Chaos from Continuous Quantum Measurements

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Abstract

The method of restricted path integrals allows one to effectively consider continuous (prolonged in time) measurements of quantum systems. Monitoring of the system coordinates is such a continuous measurement that allows one to describe a quantum system in terms of trajectories. This approach is applied to chaotic systems. The behavior of such systems is qualitatively investigated in classical and quantum regimes of the coordinate monitoring. The comparison of

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classical and quantum chaos in terms of trajectories is performed. Characteristic features of chaotic systems (observables) with respect to continuous measurements are analyzed in comparison with those of regular (non-chaotic) and quantum-nondemolition variables.

1 Introduction

Characteristic features of chaotic systems (i.e. systems showing deterministic chaos) are naturally formulated in terms of trajectories. In the framework of conventional quantum mechanics the quite different language of wave functions is used. It is usually claimed that this difference of language is unavoidable [1]. This is why the problem of quantum chaos is conventionally formulated as investigation of characteristic features (for example peculiarities of spectra) of quantum systems obtained by quantization of chaotic classical systems [1]-[3]. In the present paper theory of quantum continuous measurements (in its restricted-path-integral version [4]-[7]) is applied to investigate both classical and quantum properties of the system in terms of trajectories, i.e. with the help of one and the same language.

One may look at the subject from another point of view and ask himself how can an experimenter observe deterministic chaos.

Evidently, an experimenter should perform monitoring of some observable A of the system (during a sufficiently long time) to obtain the trajectory $[a] = \{a(t)|0 \leq t \leq T\}$. Then he should analyze statistical characteristics of this trajectory. It is important for us that an experimenter begins from performing a continuous (prolonged in time) measurement of the system.

However it is well known that any measurement of a quantum system disturbs its state. As a result the evolution of the system subject to a continuous measurement cannot be described by classical laws. Why then (and under what conditions) may one talk about deterministic chaos or chaotic character of classical trajectories? Have classical trajectories (and specifically chaotic trajectories) anything to do with reality? When usage of these trajectories is correct? What are corrections to the theory of chaos resulting from quantum features of dynamical systems? Answers to these questions can be obtained with the help of theory of continuous quantum measurements because the latter is formulated in terms of trajectories.

Of course, classical theory and particularly theory of classical chaos is

applicable in a wide class of conditions when the system may be considered as classical. However this is only an approximation. Strictly speaking, any system is a quantum one. It is important to clearly understand when classical approximation is applicable. Therefore advantageous would be such an approach which could 1) supply common language for description of both a quantum system and its classical approximation and 2) provide continuous transition from the conditions when quantum description is necessary to those when classical approximation is sufficient.

Theory of continuous quantum measurements

as2,contin[4]-[10] is an approach of this type. This may be seen first of all in the case of such a typical continuous measurement as monitoring coordinates (position) of the system. Monitoring of position gives a trajectory as its output. Therefore a quantum system undergoing position monitoring may be described in terms of trajectories, the language characteristic for classical theory.

Continuous transition from the quantum description of the system and its measurement to a classical approximation (or vice versa) may be traced when the measurement precision is continuously changed. In the case of a rough measurement quantum effects are negligible so that the classical regime of measurement is realized. When the measurement is precise enough, the quantum regime takes place with essential quantum effects.

Therefore the program may be naturally formulated to investigate position monitoring for typical chaotic systems and to analyze the distribution of the measurement results (formulated in terms of trajectories) in the case of classical and quantum regimes of measurement. The first regime should give classical chaos while the second one may be called quantum chaos. It is essential that both types of chaos will be described in this case in terms of trajectories.[1]

One of the most efficient methods for investigating continuous quantum measurements is a restricted-path-integral method [4]-[7] in which integration over all paths in the Feynman integral is replaced by integration over a restricted set of paths compatible with the measurement output. We shall apply this method to the problem in question with the aim to qualitatively

[1]It should be emphasized that the term AAquantum chaosBB will be used in this paper only in the sense AAchaos of trajectories (or rather corridors) obtained from continuous measurement performed in quantum regimeBB.

Figure 1: The output of the position monitoring may be denoted by the trajectory $[a]$ but it is adequately presented by a corridor having the width $2\Delta a$ equal to the doubled measurement error. A quantum system undergoing such a measurement should be described by the path integral with integration restricted on paths lying inside the corridor.

analyze the phenomena of quantum and classical chaos.

We shall see as a result of the analysis that there are systems showing both classical and quantum chaos in their behavior, the systems with only quantum chaos and those demonstrating neither quantum, nor classical chaos.

2 Quantum Chaos of Trajectories

The monitoring of position is an (approximate) measurement of position q in each instant of time. An output of the position monitoring may be expressed by a trajectory $[a] = \{a(t)|0 \leq t \leq T\}$. Interpretation of the measurement output is that the position q in time moment t is close to $a(t)$. Because of a finite precision Δa of the measurement, the coordinate $q(t)$ may differ from $a(t)$, but not more than by Δa. Therefore, adequate representation of the measurement output is not the trajectory $[a]$ but a corridor α of the width $2\Delta a$ centered around $[a]$ (see Fig. 1).

The main point of the restricted-path-integral method [7] is restriction of

Figure 2: The measurement is performed in the classical regime if it is rough (corridors are wide). The regime is quantum if the measurement is precise enough (in comparison with a certain quantum threshold). In the classical regime (a) only those corridors may arise as the measurement outputs which are compatible with classical predictions, i.e. contain the classical trajectory $[a_{\rm class}]$. In the quantum regime (b) corridors may be incompatible with classical predictions (demonstrating a quantum measurement noise).

the Feynman path integral on the set of paths lying inside the corridor α. This gives the probability amplitude for the given measurement output:

$$U_\alpha = \int_\alpha e^{\frac{i}{\hbar}S[q]}\, d[q]. \tag{1}$$

Monitoring of any observable A may be considered in an analogous way.

The classical regime of continuous measurement takes place when the measurement is rough enough (in comparison with some characteristic quantum threshold which should arise from the detailed calculation). This means that the corridors representing measurement outputs are wide enough. In this case only those measurement outputs have high probability which are compatible, up to the error of measurement, with the classical prediction. In the case of the position monitoring only those corridors α are probable which contain the classical trajectory $[a_{\rm class}]$ (see Fig. 2).

In the quantum regime of measurement, the measurement output may be incompatible with the classical prediction (may differ from the latter by

more than the measurement error) [7]. For example, in the case of the quantum regime of position monitoring, when corridors are narrow, even those corridors that do not contain the classical trajectory $[a_{\mathrm{class}}]$, may arise with high probability. The more precise is the measurement in this regime, the further corridors (outputs) may be from the classical trajectory.

Let us describe this in somewhat more detail taking monitoring of position (or of another observable) as an example of a continuous measurement. In this case the curve $[a]$ characterizing the measurement output, may differ from the classical trajectory $[a_{\mathrm{class}}]$ by some value δa. In the classical regime of measurement δa is equal to the measurement error Δa but it may be much more in the quantum regime. Moreover, in the quantum regime a paradoxical situation arises: the less is the measurement error Δa, the more is the variance of the measurement outputs δa. This is a typical consequence of unavoidable back reaction of the measuring device onto the measured quantum system.

This deflection of the measurement outputs from classical predictions is nothing else than the quantum measurement noise. In the present context one may call this phenomenon by *quantum chaos*.

3 Regular and Chaotic Observables

Now we can compare, from the point of view of continuous measurements, regular systems with chaotic ones.

It follows from the argument of the preceding section that, in the classical regime of measurement, the outputs coincide (up to the measurement error) with those predicted by classical theory. Therefore, if the system is regular (in the sense that deterministic chaos is absent for such a system), it behaves regular when being observed in the classical regime. Such a system has no classical chaos.[2] This is of course almost tautology. In the quantum regime of measurement such systems behave chaotic because of the quantum measurement noise (see the preceding section).

Consider now chaotic systems i.e. those that have chaotic classical trajectories. It is evident that such systems behave chaotic in the classical regime

[2] This affirmation concerns also chaotic systems if a specially chosen regular observable is measured.

of measurement (by definition of a chaotic system). What then may be said about the quantum regime?

We saw in the preceding section that, in the quantum regime, even those corridors possess high probability which are far from classical trajectories. Hence the direct connection of the measurement outputs with classical trajectories is absent. The question arises whether it is possible that no quantum chaos exists for such systems. It would take place if the high probability corridors could form some regular families leading to no chaos in laws governing these families.

However in reality this is not the case. The reason is that each corridor containing a classical trajectory, has high probability. In other words, a classical trajectory being inside the corridor is not (in the quantum regime) necessary but it is sufficient condition for the corridor having high probability. The reason is in the fact that the action functional has its extremum on a classical trajectory.

Indeed, the probability is comparatively low when the action $S[q]$ in the Feynman exponentials

$$\exp\left(\frac{i}{\hbar}S[q]\right)$$

changes quickly for $[q] \in \alpha$ so that destructive interference arises. If the variation of the action is slow for paths belonging to the given corridor α, then the exponentials sum up to give an amplitude of comparatively large absolute value. This is valid even for a corridor that contains a subset (having sufficient measure) of paths with slowly varying action. The situation is just this if the corridor has a classical trajectory inside it.[3]

One sees from this argument that each classical trajectory determines a corridor having comparatively high probability. the set of all high-probability corridors turns out to be richer than the set of classical trajectories. But classical trajectories are chaotic in the considered case of classically chaotic systems. Therefore chaos of classical trajectories results in chaos of corridors, i.e. chaos of the measurement outputs.

The conclusion that may be extracted from this argument is that a classically chaotic system (or rather a chaotic variable of such a system) possesses

[3]Of course, this statement should be formulated more correctly from the mathematical point of view and proved strictly. This is a very interesting and not easy task. However it is clear on physical ground that it is valid in typical situations.

also quantum chaos (i.e. the chaotic character of outputs of a continuous measurement performed in the quantum regime).

Summing up, we see that there are systems (observables) possessing only quantum chaos (regular systems or observables) and those possessing both classical and quantum chaos (chaotic systems or observables). It may be thought that quantum chaos is a common feature of all systems so that there exist no observable without quantum chaos. This however is not the case. We shall see in the next section that there is a class of observables (so-called quantum nondemolition, QND, observables) that possess no quantum regime of measurement and therefore no quantum chaos.

4 Quantum Nondemolition Measurements

Let us consider the physical reason for quantum measurement noise. It can be formulated as disturbing a canonically conjugate observable.

Quantum measurement noise, i.e. deflection of corridors (measurement outputs) from the classical trajectory, may be formulated as disturbing evolution of the system because of the measurement. More precisely, evolution (time dependence) of the measured observable is disturbed when it is measured in quantum regime. Why does this occur? The reason is that the measurement of an observable, because of the uncertainty principle, disturbs its canonically conjugate observable.

This may be illustrated by a simple consideration. If one measures (with a finite precision) the coordinate q of a free particle in some instant, this measurement disturbs the linear moment p. This in turn disturbs further evolution of q because of the equation of motion $m\dot{q} = p$. The same is usually valid for any pair of canonically conjugate observables.

However there are observables of a special nature (so-called quantum nondemolition, QND, observables) such that measuring them does not influence their evolution [11]-[14]. What does occur in this case? When one measures a QND observable X, its canonically conjugate observable Y is unavoidably disturbed as a result of the uncertainty principle. However a QND observable differs from a generic one in that its dynamics does not depend on the value of the canonically conjugate observable. For example, the following equation may be fulfilled for a QND variable:

$$\dot{X} = f(X). \tag{2}$$

The right-hand side of this equation does not depend of the observable Y, canonically conjugate to X. Therefore disturbance of Y, resulting from the measurement of X, does not change evolution of X.

A linear momentum p of a free particle is an example of a QND observable, since it satisfies the equation $\dot{p} = 0$. The momentum p is a QND variable even for a particle under action of external force $F(t)$ not depending on the particleBs position q (because $\dot{p} = F/m$ in this case). One more, and less trivial example is a pair of (canonically conjugate to each other) quadrature components of a harmonic oscillator:

$$\begin{aligned} X &= q \cos \omega t - \frac{p}{m\omega} \sin \omega t, \\ Y &= q \sin \omega t + \frac{p}{m\omega} \cos \omega t. \end{aligned}$$

Each of them is a QND variable.

The consequence of such a feature of QND variables is that there is no quantum regime (and therefore no quantum chaos) in their monitoring, even if an arbitrarily precise measurement is performed during the monitoring [15]. Being regular (non-chaotic), QND variables have no classical chaos too. Therefore, observables of this class possesses neither classical, nor quantum chaos. One may doubt that QND variables are necessary regular. However this may be proved in the following way.

Let X is a QND observable in a system having in general chaotic variables. May X be also chaotic or not? Being QND, the variable X has specific features. The characteristic feature of the dynamics of such a variable is that the function of time $X(t)$ is unambiguously determined by $X(0)$. This means that the equation (2) is valid. Therefore there is one-dimensional subsystem (with the coordinate X), in the system under investigation, having completely autonomous dynamics. Being one-dimensional, this subsystem cannot be chaotic, even if it is non-linear. Therefore a QND variable is necessarily regular.

Let us say several words about terminology. We discussed in detail behavior of QND observables and showed that they possess very special properties demonstrating no chaos at all. Ordinary observables (such as the coordinate of a regular system) show, as it has been discussed in the section "Quantum Chaos of Trajectories", no classical chaos, but they show quantum chaos. The variables of this class may be called (to distinguish them from QND ones) quantum demolition (QD) observables.

Table 1: Different types of observables show regular or chaotic behavior in classical and quantum regimes of observation.

	classical regime	quantum regime
QND	regular	regular
QD	regular	chaotic
SQD	chaotic	chaotic

Chaotic systems or rather chaotic observables (i.e. those showing chaotic behavior when considered classically) are in a sense opposite to the case of QND variables. Evolution of an observable of a chaotic system depends exponentially on the values of this observable and of its conjugate.[4] Therefore the measurement (performed in quantum regime) of such an observable must disturb its evolution more strongly than in the case of regular QD observables.. Observables (measurements) of chaotic systems may be called strongly quantum demolition (SQD) ones.

The chain is thus naturally determined, of QND, QD, and SQD (or chaotic) variables. QND variables are completely regular, i.e. they show no chaos in any regime of measurement. In fact no quantum regime of measurement exists for such observables. QD (usual) variables are classically regular but chaotic in the quantum regime of measurement. At last, SQD (chaotic) variables are chaotic both in quantum and classical regimes of measurement. This is illustrated by Table 1.

It is worthwhile to make one more remark. Though we talked for simplicity about chaotic observables (or even chaotic systems), in reality these observables have usually areas of regularity. In the limits of such an area the observable have all properties of a regular one. It may turn out to be ordinary (QD) observable or even QND observable while its measurement gives outputs in the regular area.

[4]This concerns generic observables of chaotic systems but this is not valid for specific regular observables existing in them (see footnote 2 above).

5 Conclusion

We have analyzed in this paper the behavior of chaotic systems from the point of view of continuous measurements with quantum effects taken into account. In most arguments monitoring of some observable A was taken as an example of continuous measurements. An output of such a measurement is presented by a trajectory $[a] = \{a(t)|0 \leq t \leq T\}$ or rather by a corridor of paths centered around the trajectory. (The quantum system undergoing the measurement is described by the Feynman path integral with integration over the corridor).

Thus even a quantum system is described by trajectories, and chaos may be described by statistics of trajectories both for quantum as well as for classical systems.

Two regimes of the measurement (classical and quantum regimes) were considered and three types of observables were distinguished as a result of the analysis: quantum nondemolition (QND), ordinary (or quantum demolition, QD), and chaotic (or strongly quantum demolition, SQD) observables.

The main goal of the paper was to found out whether chaotic behavior takes place in both considered regimes of measurement. In other words, the question was whether the system shows classical chaos and/or quantum chaos (chaos of trajectories or corridors is meant in both cases).

The answer turned out to be different for the mentioned three types of observables: QND variables have neither quantum, nor classical chaos, QD variables have only quantum chaos, and SQD variables have both classical and quantum chaos (in all these formulations quantum chaos of trajectories is meant).

It should be stressed that the analysis presented here is only preliminary and purely qualitative. Of course, much more detailed investigation is necessary. Because on nonlinearity of chaotic systems such an investigation will require numerical techniques or simulations (see [16] on the methods of such calculations).

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